Journal of Statistical Physics, Vol. 53, Nos. 5/6, 1988

Kinetic Theory of Time Correlation Functions for a Dense One-Component Plasma in a Magnetic Field

A. J. Schoolderman[1] and L. G. Suttorp[1]

Received May 2, 1988

The time-dependent correlations of a one-component plasma in a uniform magnetic field are studied with the help of kinetic theory. The time correlation functions of the particle density, the momentum density, and the kinetic energy density are evaluated for large time intervals. In the collision-dominated regime the results agree with those found from linearized magnetohydrodynamics.

KEY WORDS: Kinetic theory; time correlation functions; one-component plasma; magnetic field.

1. INTRODUCTION

Recently[1] time-dependent correlations for a one-component plasma in a magnetic field have been studied by using a linear response approach and a microscopic theory that is based on the hierarchy equations. In this way results have been obtained for the long-wavelength limit of the dynamic structure factor and for the time correlation function that describes cross correlations of the particle density and the momentum density. The time dependence of these correlations is governed by the fundamental frequencies of the so-called gyro-plasmon modes, which have been analyzed in recent years.[2–4] Since only the leading terms in the wavenumber expansion of the correlation functions are considered in ref. 1, the influence of the damping and dispersion of the collective modes on the time dependence of the correlations could not be determined. Furthermore, the treatment is confined to correlations of the particle density and the momentum density; correlations involving the energy density are not considered.

[1] Instituut voor Theoretische Fysica, Universiteit van Amsterdam, Amsterdam, The Netherlands.

An alternative approach to study time correlation functions is furnished by kinetic theory. In fact, in ref. 2 the mode spectrum of a plasma in a magnetic field has been obtained by starting from a formal kinetic equation for the one-particle time correlation function in phase space. The mode frequencies followed from the matrix elements of the memory kernel that occurs in the kinetic equation. The same method can be used as well to determine the time correlation functions of the particle density, the momentum density, and the kinetic energy density.[5,6] Indeed, for long times the collective modes will dominate these correlation functions. An advantage of the kinetic method is that dispersion and damping effects of the collective modes can easily be incorporated in the theory, at least formally.

The purpose of the present paper is to derive expressions for the time correlation functions by means of the kinetic methods described above. Furthermore, we wish to compare the results to those obtained by means of linearized magnetohydrodynamics. It is not obvious that the magnetohydrodynamic approach will lead to correct results, since the occurrence of plasma oscillations at a finite frequency may well be inconsistent with the hydrodynamic limit, which in principle implies a limit of zero frequency. Moreover, a second fundamental frequency, the Larmor frequency, shows up for a magnetized plasma. It has been shown in ref. 2 how the magnetohydrodynamic equations may nevertheless be helpful in discussing the collective mode spectrum. Its utility for the study of the time correlations remains to be assessed.

The expressions for the time correlation functions that we shall obtain in this paper are essential prerequisites for a kinetic treatment of the dynamic transport coefficients of a magnetized plasma. In fact, the long-time tails of the Green–Kubo integrands of the transport coefficients can be studied on the basis of a kinetic representation involving the time correlation functions that are the subject of this paper.

The model adopted in our treatment is the classical one-component plasma. It consists of charged particles immersed in a neutralizing inert background. The interaction between the particles and with the background is purely electrostatic. The magnetic field is supposed to be stationary and uniform in space.

After a review of the kinetic theory for a magnetized plasma in Section 2, we derive the complete set of time correlation functions in the long-wavelength limit, both from kinetic theory (Section 3) and from linearized magnetohydrodynamics (Section 4). The static limit of these functions is studied in Section 5.

2. KINETIC THEORY OF COLLECTIVE MODES

The time-dependent phase-space density correlation function C is defined by its Fourier–Laplace transform

$$C(\mathbf{k}, \mathbf{p}, \mathbf{p}', z) = -i \int_0^{\infty} dt\, e^{izt} \langle \delta f(\mathbf{k}, \mathbf{p}, t)\, \delta f(\mathbf{k}, \mathbf{p}', 0)^* \rangle \tag{2.1}$$

with $\operatorname{Im} z > 0$. The canonical-ensemble average contains the fluctuating part $\delta f = f - \langle f \rangle$ of the phase-space density

$$f(\mathbf{k}, \mathbf{p}, t) = \frac{1}{\sqrt{V}} \sum_{\alpha} \exp[-i\mathbf{k} \cdot \mathbf{r}_\alpha(t)]\, \delta[\mathbf{p} - \mathbf{p}_\alpha(t)] \tag{2.2}$$

with $\mathbf{r}_\alpha$, $\mathbf{p}_\alpha$ the position and the momentum of particle α, and V the volume of the system.

The kinetic equation for the phase-space density correlation function is an integral equation with the formal structure[2]

$$[z - \Sigma(\mathbf{k}, z)]\, C(\mathbf{k}, z) = \tilde{C}(\mathbf{k}) \tag{2.3}$$

where the momentum variables have been suppressed. Here $\tilde{C}(\mathbf{k}, \mathbf{p}, \mathbf{p}')$ is the static phase-space density correlation function $\lim_{z \to \infty} zC(\mathbf{k}, \mathbf{p}, \mathbf{p}', z)$. The kernel Σ consists of a free-streaming term φ^0, a Lorentz-force term φ^{L}, and a memory kernel φ, which is the sum of a static kernel φ^{s} and a collision kernel φ^{c}. Explicit expressions for φ^0, φ^{L}, and φ^{s} have been given in ref. 2; the collision kernel φ^{c} may be written in terms of the Liouville operator and suitable projection operators.

The time correlation functions for the particle density, the momentum density, and the kinetic energy density follow from C by writing

$$\bar{G}_{\mu\nu}(\mathbf{k}, z) = \int d\mathbf{p}\, d\mathbf{p}'\, \psi_\mu(\mathbf{p})\, C(\mathbf{k}, \mathbf{p}, \mathbf{p}', z)\, \psi_\nu(\mathbf{p}') \tag{2.4}$$

with $\mu, \nu = 0,..., 4$. Here $\psi_\mu(\mathbf{p})$ are polynomials of the momentum variable:

$$\psi_0(\mathbf{p}) = 1, \qquad \psi_i(\mathbf{p}) = \frac{p_i}{(mk_{\mathrm{B}}T)^{1/2}} \qquad (i = 1, 2, 3)$$
$$\psi_4(\mathbf{p}) = \frac{1}{\sqrt{6}} \left(\frac{p^2}{mk_{\mathrm{B}}T} - 3 \right) \tag{2.5}$$

with m the mass of the particles and T the temperature. The functions $\bar{G}_{\mu\nu}(\mathbf{k}, z)$ are related to the matrix elements of the resolvent of Σ, which are defined as

$$G_{\mu\nu}(\mathbf{k}, z) = \left\langle \mu \left| \frac{1}{z - \Sigma(\mathbf{k}, z)} \right| \nu \right\rangle \tag{2.6}$$

Here we used the notation

$$\langle \mu | A | \nu \rangle = \int d\mathbf{p}\, d\mathbf{p}'\, \psi_\mu(\mathbf{p})\, A(\mathbf{p}, \mathbf{p}')\, \psi_\nu(\mathbf{p}')\, f_0(p') \tag{2.7}$$

with $f_0(p)$ the normalized Maxwell–Boltzmann distribution. It follows from (2.3) and (2.4) that one has

$$\bar{G}_{\mu\nu}(\mathbf{k}, z) = nG_{\mu\nu}(\mathbf{k}, z)[1 + nh(k)\, \delta_{\nu 0}] \tag{2.8}$$

with n the particle density and $h(k)$ the equilibrium pair correlation function in Fourier space.

The matrix $G_{\mu\nu}(\mathbf{k}, z)$ satisfies the equation[2]

$$\sum_\lambda [z\, \delta_{\mu\lambda} - \Omega_{\mu\lambda}(\mathbf{k}, z)]\, G_{\lambda\nu}(\mathbf{k}, z) = \delta_{\mu\nu} \tag{2.9}$$

with the frequency matrix $\Omega_{\mu\nu}$ given by

$$\Omega_{\mu\nu}(\mathbf{k}, z) = \langle \mu | \Sigma | \nu \rangle + \left\langle \mu \left| \Sigma \bar{Q} \frac{1}{z - \bar{Q}\Sigma\bar{Q}} \bar{Q}\Sigma \right| \nu \right\rangle \tag{2.10}$$

Here $\bar{Q}$ is the complement of the projector $\bar{P}$, which projects a momentum-dependent function onto the space spanned by the states $|\mu\rangle$.

The frequencies of the collective modes are determined by the poles of $G_{\mu\nu}$ or $\bar{G}_{\mu\nu}$, or, equivalently, by the eigenvalues of the frequency matrix $\Omega_{\mu\nu}$. These follow from the eigenvalue equation

$$\Delta(\mathbf{k}, z) = \det[z\, \delta_{\mu\nu} - \Omega_{\mu\nu}(\mathbf{k}, z)] = 0 \tag{2.11}$$

The general form of $\Omega_{\mu\nu}(\mathbf{k}, z)$ for small k has been derived in ref. 2. In leading order of k it reads

$$\Omega_{\mu\nu}(\mathbf{k}, z) = \begin{pmatrix} 0 & v_0 k_x & v_0 k_y & v_0 k_z & 0 \\ v_0 k_D^2 k_x/k^2 & 0 & i\omega_B & 0 & v_0 k_x \bar{b}_1 \\ v_0 k_D^2 k_y/k^2 & -i\omega_B & 0 & 0 & v_0 k_y \bar{b}_1 \\ v_0 k_D^2 k_z/k^2 & 0 & 0 & 0 & v_0 k_z(\bar{b}_1 + \bar{b}_2) \\ 0 & v_0 k_x \bar{b}_1 & v_0 k_y \bar{b}_1 & v_0 k_z(\bar{b}_1 + \bar{b}_2) & -ic \end{pmatrix} \tag{2.12}$$

with $v_0 = (k_B T/m)^{1/2}$ the thermal velocity, $k_D = (ne^2/k_B T)^{1/2}$ the Debye wavenumber, and $\omega_B = eB/mc$ the Larmor frequency in the magnetic field with strength B. The direction of the magnetic field $\mathbf{B}$ is chosen parallel to the positive z axis. Furthermore, $\bar{b}_i(z)$ and $c(z)$ are dynamical coefficients, which are defined by writing, for small wavenumber,

$$\langle i|\varphi^0+\varphi^c|4\rangle + \left\langle i \left| \Sigma\bar{Q} \frac{1}{z-\bar{Q}\Sigma\bar{Q}} \bar{Q}\Sigma \right| 4 \right\rangle$$
$$= v_0 k[\hat{k}_i \bar{b}_1(z) + \hat{B}_i \hat{k}_{\|} \bar{b}_2(z)] \qquad (i=1, 2, 3) \tag{2.13}$$

$$\langle 4|\varphi^c|4\rangle + \left\langle 4 \left| \Sigma\bar{Q} \frac{1}{z-\bar{Q}\Sigma\bar{Q}} \bar{Q}\Sigma \right| 4 \right\rangle$$
$$= -ic(z) - iv_0^2 k^2[d_1(z) + \hat{k}_{\|}^2\, d_2(z)] \tag{2.14}$$

up to first and second order in k, respectively. To obtain these expressions, one uses the cylinder symmetry of the system; $\hat{\mathbf{B}}$ and $\hat{\mathbf{k}}$ are unit vectors in the direction of the magnetic field and the wave vector, respectively, and $\hat{k}_{\|} = \hat{\mathbf{k}} \cdot \hat{\mathbf{B}}$ is the component of $\hat{\mathbf{k}}$ in the direction of the magnetic field. For vanishingly small z the coefficients $\bar{b}_i(z)$ and $c(z)$ satisfy the limiting relations

$$\lim_{z \to i0} \bar{b}_1(z) = \left(\frac{2}{3}\right)^{1/2} \frac{1}{nk_B} \left(\frac{\partial P}{\partial T}\right)_n \tag{2.15}$$

$$\lim_{z \to i0} \bar{b}_2(z) = 0 \tag{2.16}$$

$$\lim_{z \to i0} \frac{c(z)}{z} = i\left(1 - \frac{2c_V}{3k_B}\right) \tag{2.17}$$

with P the equilibrium pressure and c_V the isochoric specific heat per particle.

The determinant $\Delta(\mathbf{k}, z)$ occurring in the eigenvalue equation (2.11) is found to factorize, up to second order in k, in the following way:

$$\Delta(\mathbf{k}, z) = \frac{2c_V}{3k_B} [z - z_T(z)] \prod_{\lambda\rho} [z - z_{\lambda\rho}(z)] \tag{2.18}$$

The functions $z_T(z)$ and $z_{\lambda\rho}(z)$, with $\lambda = \pm 1$, $\rho = \pm 1$, determine the mode frequencies of the heat mode and of the four gyro-plasmon modes. The function $z_T(z)$ has the form

$$z_T(z) = \left(1 - \frac{3k_B}{2c_V}\right) z - i\frac{3k_B}{2c_V} \{c(z) + v_0^2 k^2[d_1(z) + \hat{k}_{\|}^2\, d_2(z)]\} \tag{2.19}$$

Because of (2.17) one finds for vanishing wavenumber as a solution of $z_T(z) = z$ the trivial result $z = 0$. Up to second order of k the mode frequency is hence obtained by substituting $z = i0$ in (2.19), so that we get the mode frequency

$$z_T \equiv z_T(i0) = -\frac{i}{nc_V} k^2(\lambda_\perp \hat{k}_\perp^2 + \lambda_{\|} \hat{k}_{\|}^2) \tag{2.20}$$

with $\hat{\mathbf{k}}_\perp = \hat{\mathbf{k}} - \hat{k}_{\|} \hat{\mathbf{B}}$ the transverse component of $\hat{\mathbf{k}}$, and $\lambda_\perp$ and $\lambda_{\|}$ the transverse and the longitudinal heat conduction coefficients

$$\lambda_\perp = \tfrac{3}{2}(nk_B^2 T/m)\, d_1(i0) \tag{2.21}$$

$$\lambda_{\|} = \tfrac{3}{2}(nk_B^2 T/m)[d_1(i0) + d_2(i0)] \tag{2.22}$$

The fundamental gyro-plasmon frequencies are given by ρw_λ, where w_λ is defined as

$$w_\lambda = \tfrac{1}{2}(\omega_p^2 + \omega_B^2 + 2\omega_p \omega_B \hat{k}_{\|})^{1/2} + \tfrac{1}{2}\lambda(\omega_p^2 + \omega_B^2 - 2\omega_p \omega_B \hat{k}_{\|})^{1/2} \tag{2.23}$$

with $\omega_p = (ne^2/m)^{1/2}$ the plasma frequency. These frequencies satisfy the identity

$$w_\lambda^4 - w_\lambda^2(\omega_p^2 + \omega_B^2) + \omega_p^2 \omega_B^2 \hat{k}_{\|}^2 = 0 \tag{2.24}$$

Up to second order in k the functions $z_{\lambda\rho}(z)$ have the form

$$z_{\lambda\rho}(z) = \rho w_\lambda - ik^2 D_{\lambda\rho}(z) \tag{2.25}$$

The damping and dispersion functions read

$$D_{\lambda\rho}(z) = \frac{iv_0^2}{2w_\lambda^2(2w_\lambda^2 - \omega_p^2 - \omega_B^2)} \left[N_{1\lambda}^{\rho}(z) + \frac{N_{2\lambda}(z)}{z + ic(z)} \right] \tag{2.26}$$

The functions $N_{1\lambda}^{\rho}(z)$ and $N_{2\lambda}(z)$, the explicit form of which is given in the Appendix, depend on the dynamic coefficients $\bar{b}_i(z)$ and on the seven dynamic viscosity coefficients of the magnetized plasma. The frequencies of the gyro-plasmon modes up to order k^2 follow by substituting $z = \rho w_\lambda$ in (2.25):

$$z_{\lambda\rho} \equiv z_{\lambda\rho}(\rho w_\lambda) = \rho w_\lambda - ik^2 D_{\lambda\rho} \tag{2.27}$$

with $D_{\lambda\rho} \equiv D_{\lambda\rho}(\rho w_\lambda)$.

3. DERIVATION OF TIME CORRELATION FUNCTIONS FROM KINETIC THEORY

The particle-density time correlation function (t.c.f.), or dynamic structure factor, is defined as

$$S^{nn}(\mathbf{k}, z) = n^{-1} \int d\mathbf{p}\, d\mathbf{p}'\, C(\mathbf{k}, \mathbf{p}, \mathbf{p}', z) = [1 + nh(k)]\, G_{00}(\mathbf{k}, z) \tag{3.1}$$

The dynamic structure factor depends on the wave vector $\mathbf{k}$ only through its norm k and its component $k_{||}$ along the magnetic field, since the plasma is cylinder-symmetric. Additional constraints are obtained by employing parity invariance and the invariance under the combined effect of time reversal and of time translation. These symmetries imply

$$C(\mathbf{k}, \mathbf{p}, \mathbf{p}', \hat{\mathbf{B}}, z) = C(-\mathbf{k}, -\mathbf{p}, -\mathbf{p}', \hat{\mathbf{B}}, z) \tag{3.2}$$

$$C(\mathbf{k}, \mathbf{p}, \mathbf{p}', \hat{\mathbf{B}}, z) = C(\mathbf{k}, \mathbf{p}', \mathbf{p}, -\hat{\mathbf{B}}, z) \tag{3.3}$$

where the dependence on the direction of the field has been rendered explicit. From these relations it follows that S^{nn} is an even function of $k_{||}$, or $\hat{k}_{||} = k_{||}/k$.

The matrix element G_{00} is obtained from (2.9) as

$$G_{00}(\mathbf{k}, z) = \frac{M_{00}(\mathbf{k}, z)}{\Delta(\mathbf{k}, z)} \tag{3.4}$$

with $M_{00}(\mathbf{k}, z)$ the minor determinant associated with the $(0, 0)$ element in the determinant $\Delta(\mathbf{k}, z)$ defined in (2.11). Using (2.12), we find

$$M_{00}(\mathbf{k}, z) = \frac{2c_V}{3k_{\mathrm{B}}} z(z^2 - \omega_B^2)[z - z_T(z)] \tag{3.5}$$

in lowest order in k. Inserting this expression and (2.18) into (3.1) with (3.4), we arrive at

$$S^{nn}(\mathbf{k}, z) = [1 + nh(k)] \frac{z(z^2 - \omega_B^2)}{\prod_{\lambda\rho} [z - z_{\lambda\rho}(z)]} \tag{3.6}$$

It should be noted that the thermal mode has dropped out. We have retained terms up to order k^2 in the denominator, so that the positions of the poles in the complex z plane are correctly given by (3.6) up to that order.

For large values of t the dynamic structure factor $S^{nn}(\mathbf{k}, t)$ is dominated by the contribution of the poles that are closest to the real axis. These are the gyro-plasmon mode poles situated at $z = z_{\lambda\rho}(\rho w_\lambda)$. Evaluating the residues, we obtain in the "pole approximation" the following asymptotic form for $S^{nn}(\mathbf{k}, t)$:

$$S^{nn}(\mathbf{k}, t) = \frac{1}{2} \frac{k^2}{k_{\mathrm{D}}^2} \sum_{\lambda\rho} \frac{w_\lambda^2 - \omega_B^2}{2w_\lambda^2 - \omega_p^2 - \omega_B^2} e^{-iz_{\lambda\rho}t} \tag{3.7}$$

where we used that $1 + nh(k) = k^2/k_{\mathrm{D}}^2$ in leading order of the wavenumber. The amplitude in front of the exponential function is given here only in lowest order of k; higher-order terms are considered in the Appendix. Since t gets large, it is expedient to retain nevertheless terms up to order k^2, as given in (2.27), in the exponent.

The expression (3.6) may also be employed to study the dynamical structure factor for arbitrary finite t. To that end, we drop all terms of higher order in the denominator, so that it gets the simple form $\prod_{\lambda\rho}(z-\rho w_\lambda)$. Upon using a partial fraction decomposition and performing the inverse Fourier transform, we get the expression

$$S^{nn}(\mathbf{k}, t)=\frac{1}{2}\frac{k^2}{k_\mathrm{D}^2}\sum_{\lambda\rho}\frac{w_\lambda^2-\omega_B^2}{2w_\lambda^2-\omega_p^2-\omega_B^2}e^{-i\rho w_\lambda t} \tag{3.8}$$

which gives the structure factor for arbitrary t in lowest order in k. Hence, in the long-wavelength limit the dynamic structure factor is governed by the gyro-plasmon modes for all t. The result obtained here agrees with that derived in ref. 1.

The time-dependent cross correlations of the particle density and the components of the momentum density are analyzed conveniently by introducing three reduced t.c.f.'s in the following way:

$$\begin{aligned}\int d\mathbf{p}\, d\mathbf{p}'\, \mathbf{p}'\, C(\mathbf{k}, \mathbf{p}, \mathbf{p}', z)\\ = \mathbf{k}_{\|} S_{\|}^{ng}(\mathbf{k}, z)+\mathbf{k}_{\perp} S_{\perp}^{ng}(\mathbf{k}, z)+\mathbf{k}\wedge\hat{\mathbf{B}} S_t^{ng}(\mathbf{k}, z)\end{aligned} \tag{3.9}$$

with $\mathbf{k}_{\|}=\mathbf{k}\cdot\hat{\mathbf{B}}\hat{\mathbf{B}}$ and $\mathbf{k}_\perp=\mathbf{k}-\mathbf{k}_{\|}$. The functions S_i^{ng} depend on k, $\hat{k}_{\|}$, and z; parity invariance implies that they are even in $\hat{k}_{\|}$. Using (3.3), we can derive an analogous expression for $\int d\mathbf{p}\, d\mathbf{p}'\, \mathbf{p}C$; it differs from (3.9) by a minus sign in front of the last term.

Kinetic expressions for the reduced t.c.f.'s are obtained straightforwardly with the help of (2.9), (2.12), and (2.18). In leading order of the wavenumber we get, upon employing the pole approximation, the asymptotic expression

$$S_i^{ng}(\mathbf{k}, t)=\frac{1}{2}nk_\mathrm{B}T\sum_{\lambda\rho}\frac{\rho f_{i,\lambda\rho}}{w_\lambda(2w_\lambda^2-\omega_p^2-\omega_B^2)}e^{-iz_{\lambda\rho}t} \tag{3.10}$$

with the abbreviations

$$f_{\|,\lambda\rho}=w_\lambda^2-\omega_B^2 \tag{3.11}$$

$$f_{\perp,\lambda\rho}=w_\lambda^2 \tag{3.12}$$

$$f_{t,\lambda\rho}=-i\omega_B\rho w_\lambda \tag{3.13}$$

As before, the heat mode does not contribute in lowest order of k. Expressions for S_i^{ng} that are valid for arbitrary t and for vanishing wavenumber can be derived on a par with (3.8). These differ from (3.10) in containing ρw_λ instead of $z_{\lambda\rho}$ in the exponential function. Substitution in (3.9) yields a result that corroborates the findings of ref. 1. In the Appendix the t.c.f.'s S_i^{ng} are studied up to order k^2.

The expressions (3.7) and (3.10) are connected by an identity that follows from particle conservation. In fact, since

$$\frac{\partial}{\partial t}\int d\mathbf{p}\, d\mathbf{p}'\, C(\mathbf{k}, \mathbf{p}, \mathbf{p}', t) = -i\int d\mathbf{p}\, d\mathbf{p}'\,\frac{\mathbf{k}\cdot\mathbf{p}'}{m}\, C(\mathbf{k}, \mathbf{p}, \mathbf{p}', t) \qquad (3.14)$$

one has the relation

$$imn\frac{\partial}{\partial t} S^{nn}(\mathbf{k}, t) = k_{\parallel}^2 S_{\parallel}^{ng}(\mathbf{k}, t) + k_{\perp}^2 S_{\perp}^{ng}(\mathbf{k}, t) \qquad (3.15)$$

which is easily checked to be satisfied by (3.7) and (3.10).

The correlations between the components of the momentum density can be expressed in terms of reduced t.c.f.'s that are introduced by writing a linear combination of all independent second-rank tensors depending on $\hat{\mathbf{k}}$ and $\hat{\mathbf{B}}$. Using the properties (3.2) and (3.3), this linear combination can be cast into the form

$$\begin{aligned}&\int d\mathbf{p}\, d\mathbf{p}'\, \mathbf{p}\mathbf{p}' C(\mathbf{k}, \mathbf{p}, \mathbf{p}', z)\\ &\quad= \mathbf{U} S_1^{gg}(\mathbf{k}, z) + \hat{\mathbf{k}}_{\parallel}\hat{\mathbf{k}}_{\parallel} S_2^{gg}(\mathbf{k}, z)\\ &\qquad+ (\hat{\mathbf{k}}_{\parallel}\hat{\mathbf{k}}_{\perp} + \hat{\mathbf{k}}_{\perp}\hat{\mathbf{k}}_{\parallel})\, S_3^{gg}(\mathbf{k}, z) + \hat{\mathbf{k}}_{\perp}\hat{\mathbf{k}}_{\perp} S_4^{gg}(\mathbf{k}, z)\\ &\quad+ \boldsymbol{\varepsilon}\cdot\hat{\mathbf{k}}_{\perp}\hat{k}_{\parallel} S_5^{gg}(\mathbf{k}, z) + \boldsymbol{\varepsilon}\cdot\hat{\mathbf{B}} S_6^{gg}(\mathbf{k}, z)\end{aligned} \qquad (3.16)$$

with $\hat{\mathbf{k}}_{\parallel} = \hat{k}_{\parallel}\hat{\mathbf{B}}$, $\boldsymbol{\varepsilon}$ the Levi-Civita tensor, and $\mathbf{U}$ the unit tensor. The functions S_i^{gg} depend on k, $\hat{k}_{\parallel}$, and z, and are even in $\hat{k}_{\parallel}$. The kinetic expressions for these reduced t.c.f.'s that follow from (2.9), (2.12), and (2.18) read in the pole approximation and in leading order of k

$$S_i^{gg}(\mathbf{k}, t) = \frac{1}{2} nmk_B T \sum_{\lambda\rho} \frac{g_{i,\lambda\rho}}{2w_\lambda^2 - \omega_p^2 - \omega_B^2}\, e^{-iz_{\lambda\rho}t} \qquad (3.17)$$

with the coefficients

$$g_{1,\lambda\rho} = w_\lambda^2 - \omega_p^2 \qquad (3.18)$$

$$g_{2,\lambda\rho} = \omega_p^2[w_\lambda^4 - w_\lambda^2(\omega_p^2 + \omega_B^2) + \omega_B^4]/[w_\lambda^2(w_\lambda^2 - \omega_p^2 - \omega_B^2)] \qquad (3.19)$$

$$g_{3,\lambda\rho} = g_{4,\lambda\rho} = \omega_p^2 \tag{3.20}$$

$$g_{5,\lambda\rho} = -i\rho\omega_B\omega_p^2/w_\lambda \tag{3.21}$$

$$g_{6,\lambda\rho} = i\rho w_\lambda(w_\lambda^2 - \omega_p^2)/\omega_B \tag{3.22}$$

Again the contribution of the heat mode is of higher order in k. Expressions valid for all t and $k=0$ follow from (3.17) by replacing $z_{\lambda\rho}$ by ρw_λ in the exponential. In the Appendix the t.c.f.'s S_i^{gg} are studied up to order k^2.

The conservation of particle number gives rise to three relations among the t.c.f.'s (3.10) and (3.17):

$$im\frac{\partial}{\partial t} S_{\parallel}^{ng}(\mathbf{k}, t) = S_1^{gg}(\mathbf{k}, t) + \hat{k}_{\parallel}^2 S_2^{gg}(\mathbf{k}, t) + \hat{k}_{\perp}^2 S_3^{gg}(\mathbf{k}, t) \tag{3.23}$$

$$im\frac{\partial}{\partial t} S_{\perp}^{ng}(\mathbf{k}, t) = S_1^{gg}(\mathbf{k}, t) + \hat{k}_{\parallel}^2 S_3^{gg}(\mathbf{k}, t) + \hat{k}_{\perp}^2 S_4^{gg}(\mathbf{k}, t) \tag{3.24}$$

$$im\frac{\partial}{\partial t} S_t^{ng}(\mathbf{k}, t) = \hat{k}_{\parallel}^2 S_5^{gg}(\mathbf{k}, t) - S_6^{gg}(\mathbf{k}, t) \tag{3.25}$$

Let us turn now to the correlation functions involving the kinetic energy density. Since the potential energy is not a one-particle property, the kinetic approach used here cannot be employed to obtain correlation functions for the total energy density. The cross correlations of the kinetic energy density and the particle density are described by the t.c.f.

$$S^{\varepsilon^{\mathrm{kin}} n}(\mathbf{k}, z) = \int d\mathbf{p}\, d\mathbf{p}' \frac{p^2}{2m} C(\mathbf{k}, \mathbf{p}, \mathbf{p}', z) \tag{3.26}$$

It can be expressed in terms of the matrix elements $G_{\mu\nu}$ of the resolvent of Σ:

$$S^{\varepsilon^{\mathrm{kin}} n}(\mathbf{k}, z) = \frac{3}{2} n k_{\mathrm{B}} T[1 + nh(k)] \left[G_{00}(\mathbf{k}, z) + \left(\frac{2}{3}\right)^{1/2} G_{40}(\mathbf{k}, z) \right] \tag{3.27}$$

Evaluating G_{40} in the same way as G_{00} in (3.4)–(3.5), we get up to second order in k

$$S^{\varepsilon^{\mathrm{kin}} n}(\mathbf{k}, z) = \frac{3}{2} n k_{\mathrm{B}} T \frac{k^2}{k_{\mathrm{D}}^2} \left\{ \frac{z(z^2 - \omega_B^2)}{\prod_{\lambda\rho}[z - z_{\lambda\rho}(z)]} + \left(\frac{2}{3}\right)^{1/2} \frac{3k_{\mathrm{B}}}{2c_V} \omega_p^2 \frac{\bar{b}_1(z)(z^2 - \omega_B^2 \hat{k}_{\parallel}^2) + \bar{b}_2(z)(z^2 - \omega_B^2)\hat{k}_{\parallel}^2}{\prod_{\lambda\rho}[z - z_{\lambda\rho}(z)][z - z_T(z)]} \right\} \tag{3.28}$$

The asymptotic expression for $S^{\varepsilon^{\mathrm{kin}}n}(\mathbf{k}, t)$ valid for large t is obtained by using the pole approximation as before. In this way we find

$$S^{\varepsilon^{\mathrm{kin}}n}(\mathbf{k}, t) = \frac{3}{2} n k_{\mathrm{B}} T \frac{k^2}{k_{\mathrm{D}}^2} \left\{ -\frac{1}{n c_V} \left(\frac{\partial P}{\partial T} \right)_n e^{-i z_T t} + \frac{1}{2} \sum_{\lambda\rho} \frac{w_\lambda^2 - \omega_B^2}{2 w_\lambda^2 - \omega_p^2 - \omega_B^2} e^{-i z_{\lambda\rho} t} + \frac{1}{2} \left(\frac{2}{3} \right)^{1/2} \sum_{\lambda\rho} \frac{\rho w_\lambda (w_\lambda^2 - \omega_B^2)}{\omega_B^2 (2 w_\lambda^2 - \omega_p^2 - \omega_B^2)} \times \frac{\omega_B^2 \bar{b}_1(\rho w_\lambda) - (w_\lambda^2 - \omega_p^2 - \omega_B^2)\, \bar{b}_2(\rho w_\lambda)}{\rho w_\lambda + i c(\rho w_\lambda)} e^{-i z_{\lambda\rho} t} \right\} \tag{3.29}$$

We have employed (2.15) and (2.16) to evaluate $\bar{b}_i(z)$ at $z = i0$, as it occurred in the heat mode contribution. The gyro-plasmon mode contributions contain these dynamic coefficients, and $c(z)$ as well, at the finite frequency $z = \rho w_\lambda$. Since $\bar{b}_i(z)$ and $c(z)$ are not known explicitly for arbitrary z, it is not possible to establish a general expression for $S^{\varepsilon^{\mathrm{kin}}n}(\mathbf{k}, t)$ that is valid for arbitrary t, not even in leading order in k.

To analyze the correlations between the kinetic energy density and the components of the momentum density, we introduce reduced t.c.f.'s as in (3.9):

$$\int d\mathbf{p}\, d\mathbf{p}' \frac{p^2}{2m} \mathbf{p}' C(\mathbf{k}, \mathbf{p}, \mathbf{p}', z) = \mathbf{k}_{\|} S_{\|}^{\varepsilon^{\mathrm{kin}}g}(\mathbf{k}, z) + \mathbf{k}_{\perp} S_{\perp}^{\varepsilon^{\mathrm{kin}}g}(\mathbf{k}, z) + \mathbf{k} \wedge \hat{\mathbf{B}} S_t^{\varepsilon^{\mathrm{kin}}g}(\mathbf{k}, z) \tag{3.30}$$

The left-hand side is a linear combination of the matrix elements G_{0i} and G_{4i} $(i = 1, 2, 3)$. Upon evaluating these matrix elements with the help of (2.12), we get for small k

$$S_i^{\varepsilon^{\mathrm{kin}}g}(\mathbf{k}, z) = \frac{3}{2} n (k_{\mathrm{B}} T)^2 \left\{ \frac{h_i'(z)}{\prod_{\lambda\rho} [z - z_{\lambda\rho}(z)]} + \left(\frac{2}{3} \right)^{1/2} \frac{3 k_{\mathrm{B}}}{2 c_V} \frac{h_i''(z)}{\prod_{\lambda\rho} [z - z_{\lambda\rho}(z)][z - z_T(z)]} \right\} \tag{3.31}$$

with

$$h_{\|}' = z^2 - \omega_B^2, \qquad h_{\perp}' = z^2, \qquad h_t' = -i\omega_B z \tag{3.32}$$

and

$$h''_{\|} = \bar{b}_1(z)\, z(z^2 - \omega_B^2) + \bar{b}_2(z)\, z(z^2 - \omega_B^2 - \omega_p^2 \hat{k}_{\perp}^2) \tag{3.33}$$

$$h''_{\perp} = \bar{b}_1(z) z^3 + \bar{b}_2(z) z \omega_p^2 \hat{k}_{\|}^2 \tag{3.34}$$

$$h''_t = -i\omega_B h''_{\perp}/z \tag{3.35}$$

The t.c.f.'s for large t follow by using again the pole approximation. The results are, in lowest order of k,

$$S_i^{\varepsilon^{\text{kin}} g}(\mathbf{k}, t) = \frac{3}{4} n(k_B T)^2 \sum_{\lambda\rho} f_{i,\lambda\rho} \left\{ \frac{\rho}{w_\lambda(2w_\lambda^2 - \omega_p^2 - \omega_B^2)} + \left(\frac{2}{3}\right)^{1/2} \frac{\omega_B^2 \bar{b}_1(\rho w_\lambda) - (w_\lambda^2 - \omega_p^2 - \omega_B^2)\, \bar{b}_2(\rho w_\lambda)}{\omega_B^2(2w_\lambda^2 - \omega_p^2 - \omega_B^2)[\rho w_\lambda + ic(\rho w_\lambda)]} \right\} e^{-iz_{\lambda\rho} t} \tag{3.36}$$

with the same $f_{i,\lambda\rho}$ as in (3.11)–(3.13). The heat mode contributions are of higher order in k, since the residues are proportional to z_T. As before, the evaluation of the t.c.f.'s for arbitrary t is hampered by the lack of knowledge on the dynamic coefficients $\bar{b}_i(z)$ and $c(z)$ as functions of the complex frequency z.

Because of particle conservation, the t.c.f.'s given in (3.29) and (3.36) satisfy the relation

$$im \frac{\partial}{\partial t} S^{\varepsilon^{\text{kin}} n}(\mathbf{k}, t) = k_{\|}^2 S_{\|}^{\varepsilon^{\text{kin}} g}(\mathbf{k}, t) + k_{\perp}^2 S_{\perp}^{\varepsilon^{\text{kin}} g}(\mathbf{k}, t) \tag{3.37}$$

The heat mode contribution in (3.29) drops out upon taking the derivative, since it leads to a term of higher order in k.

Finally, we consider the autocorrelations of the kinetic energy density, which are determined by the t.c.f.

$$S^{\varepsilon^{\text{kin}} \varepsilon^{\text{kin}}}(\mathbf{k}, z) = \int d\mathbf{p}\, d\mathbf{p}' \frac{p^2}{2m} \frac{p'^2}{2m} C(\mathbf{k}, \mathbf{p}, \mathbf{p}', z) \tag{3.38}$$

Expressing it in terms of the matrix elements $G_{\mu\nu}$ and evaluating these with the help of (2.12), we find

$$S^{\varepsilon^{\text{kin}} \varepsilon^{\text{kin}}}(\mathbf{k}, z) = \frac{9}{4} n(k_B T)^2 \frac{k_B}{c_V} \frac{z^2(z^2 - \omega_p^2 - \omega_B^2) + \omega_p^2 \omega_B^2 \hat{k}_{\|}^2}{\prod_{\lambda\rho}[z - z_{\lambda\rho}(z)][z - z_T(z)]} \tag{3.39}$$

for small k. The pole approximation yields the asymptotic expression valid for large t:

$$S^{\varepsilon^{\text{kin}} \varepsilon^{\text{kin}}}(\mathbf{k}, t) = \frac{9}{4} n(k_B T)^2 \frac{k_B}{c_V} e^{-iz_T t} \tag{3.40}$$

The gyro-plasmon modes drop out because of (2.24).

The complete set of t.c.f.'s that are accessible in a straightforward way by means of kinetic theory has now been derived. We have found that some of these, namely those describing correlations between the kinetic energy density on one hand and the particle density or the momentum density on the other, contain dynamic coefficients with finite frequencies already in leading order in k, whereas the remaining t.c.f.'s do not depend on these coefficients in lowest order in the wavenumber. If higher terms in the expansion in powers of k are included, dynamic coefficients show up in all t.c.f.'s, as is shown in the Appendix.

In a strongly coupled plasma the collisions are expected to dominate the collective behavior. In that case both the plasma frequency and the Larmor frequency are small compared to the collision frequency. The coefficients $\bar{b}_i(z)$ and $c(z)$ may then be assumed to vary slowly for frequencies up to w_λ, so that both can be replaced by their values near zero frequency. Using (2.15)–(2.17), we obtain for the collision-dominated plasma the t.c.f.'s (3.29) and (3.36):

$$S^{\varepsilon^{\text{kin}}n}(\mathbf{k}, t) = \frac{3}{2} n k_{\text{B}} T \frac{k^2}{k_{\text{D}}^2} \left\{ -\frac{1}{n c_V} \left(\frac{\partial P}{\partial T} \right)_n e^{-iz_T t} \right.$$
$$\left. + \frac{1}{2} \left[\frac{1}{n c_V} \left(\frac{\partial P}{\partial T} \right)_n + 1 \right] \sum_{\lambda\rho} \frac{w_\lambda^2 - \omega_B^2}{2w_\lambda^2 - \omega_p^2 - \omega_B^2} e^{-iz_{\lambda\rho} t} \right\} \quad (3.41)$$

$$S_i^{\varepsilon^{\text{kin}}g}(\mathbf{k}, t) = \frac{3}{4} n (k_{\text{B}} T)^2 \left[\frac{1}{n c_V} \left(\frac{\partial P}{\partial T} \right)_n + 1 \right]$$
$$\times \sum_{\lambda\rho} \frac{\rho f_{i,\lambda\rho}}{w_\lambda (2w_\lambda^2 - \omega_p^2 - \omega_B^2)} e^{-iz_{\lambda\rho} t} \quad (3.42)$$

Thermodynamic quantities have taken the place of the dynamic coefficients in these expressions. We shall show presently how these results can be derived from magnetohydrodynamics as well.

4. DERIVATION OF TIME CORRELATION FUNCTIONS FROM MAGNETOHYDRODYNAMICS

The linearized magnetohydrodynamic equations for a one-component plasma in a uniform static magnetic field read

$$\frac{\partial}{\partial t} \delta n(\mathbf{r}, t) + n \nabla \cdot \mathbf{v}(\mathbf{r}, t) = 0 \quad (4.1)$$

$$nm \frac{\partial}{\partial t} \mathbf{v}(\mathbf{r}, t) + \nabla \delta P(\mathbf{r}, t) - \nabla \cdot \boldsymbol{\eta} : \nabla \mathbf{v}(r, t)$$
$$= ne\mathbf{E}(\mathbf{r}, t) + nm\omega_B \mathbf{v}(\mathbf{r}, t) \wedge \hat{\mathbf{B}} \quad (4.2)$$

$$\frac{\partial}{\partial t}\,\delta\varepsilon(\mathbf{r},t)+(\varepsilon+P)\,n\,\nabla\cdot\mathbf{v}(\mathbf{r},t)$$
$$-\frac{1}{nc_V}\nabla\cdot\boldsymbol{\lambda}\cdot\nabla\left[\delta\varepsilon(\mathbf{r},t)+\left(\frac{3}{2}k_BT-\frac{3}{n\kappa_T}\right)\delta n(\mathbf{r},t)\right]=0 \qquad (4.3)$$

The local fluctuations of the particle density n, the hydrodynamic pressure P, and the energy density ε have been written as $\delta n(\mathbf{r}, t)$, $\delta P(\mathbf{r}, t)$, and $\delta\varepsilon(\mathbf{r}, t)$, respectively. The hydrodynamic velocity is denoted by $\mathbf{v}(\mathbf{r}, t)$. The local electric field $\mathbf{E}$ satisfies the Maxwell equation

$$\nabla\cdot\mathbf{E}(\mathbf{r},t)=e\,\delta n(\mathbf{r},t) \qquad (4.4)$$

with e the particle charge. Furthermore, κ_T is the isothermal compressibility. The structure of the thermal conductivity tensor $\boldsymbol{\lambda}$ and of the viscosity tensor $\boldsymbol{\eta}$ has been discussed in ref. 2.

To derive the t.c.f.'s, a Fourier–Laplace transform is applied to Eqs. (4.1)–(4.3). The determinant of the resulting set of linear equations for $\delta n(\mathbf{k}, z)$, $\mathbf{v}(\mathbf{k}, z)$, and $\delta\varepsilon(\mathbf{k}, z)$ determines the magnetohydrodynamic collective modes. As in kinetic theory, the mode spectrum consists of one heat mode and four gyro-plasmon modes. In the present case the coefficients of damping and dispersion of the gyro-plasmon modes depend on static magnetohydrodynamic transport coefficients.

The time-dependent correlations of the particle density fluctuations δn, the momentum density $\mathbf{g}=nm\mathbf{v}$, and the energy density fluctuations $\delta\varepsilon$ follow by solving the linear equations for these quantities in terms of their initial values at $t=0$ and using the static correlation formulas, which have been established previously.[7,8] The determinant of the set of linear equations is a fifth-order polynomial in z, with frequency-independent coefficients. As a consequence, a partial fraction decomposition may be employed to write the magnetohydrodynamic t.c.f.'s as linear combinations of contributions from the five collective modes. The ensuing expressions can be compared to the kinetic results of the previous section.

In leading order of the wavenumber the t.c.f.'s involving the particle density and the momentum density only can be obtained straightforwardly by solving the reduced set of equations that follow from (4.1)–(4.4) by discarding terms of higher order in k. In Fourier language this set reads

$$\frac{\partial}{\partial t}\,\delta n(\mathbf{k},t)=-in\mathbf{k}\cdot\mathbf{v}(\mathbf{k},t) \qquad (4.5)$$

$$nm\frac{\partial}{\partial t}\,\mathbf{v}(\mathbf{k},t)=-ine^2(\mathbf{k}/k^2)\,\delta n(\mathbf{k},t)+nm\omega_B\mathbf{v}(\mathbf{k},t)\wedge\hat{\mathbf{B}} \qquad (4.6)$$

Solving these equations, we obtain δn and $\mathbf{v}$ as linear combinations of the gyro-plasmon modes:

$$\frac{k_{\mathrm{D}}}{k\sqrt{n}}\,\delta n(\mathbf{k},t)=\sum_{\lambda\rho} a_{\lambda\rho}e^{-i\rho w_\lambda t} \tag{4.7}$$

$$\left(\frac{nm}{k_{\mathrm{B}}T}\right)^{1/2}\mathbf{v}(\mathbf{k},t)=\sum_{\lambda\rho} a_{\lambda\rho}\,\mathbf{v}^*_{\lambda\rho}e^{-i\rho w_\lambda t} \tag{4.8}$$

with the auxiliary vector[3]

$$\mathbf{v}_{\lambda\rho}=\frac{\rho w_\lambda \omega_p}{w_\lambda^2-\omega_B^2}\,\hat{\mathbf{k}}_\perp+\frac{\rho\omega_p}{w_\lambda}\,\hat{\mathbf{k}}_{\|}-\frac{i\omega_p\omega_B}{w_\lambda^2-\omega_B^2}\,\hat{\mathbf{k}}\wedge\hat{\mathbf{B}} \tag{4.9}$$

and with coefficients $a_{\lambda\rho}$ that are determined by the initial conditions. Using the relations

$$\mathbf{v}^*_{\lambda\rho}\cdot\mathbf{v}_{\lambda'\rho'}=-1+\frac{2(2w_\lambda^2-\omega_p^2-\omega_B^2)}{w_\lambda^2-\omega_B^2}\,\delta_{\lambda\lambda'}\,\delta_{\rho\rho'} \tag{4.10}$$

we easily find

$$a_{\lambda\rho}=\frac{w_\lambda^2-\omega_B^2}{2(2w_\lambda^2-\omega_p^2-\omega_B^2)}\left[\frac{k_{\mathrm{D}}}{k\sqrt{n}}\,\delta n(\mathbf{k},0)+\left(\frac{nm}{k_{\mathrm{B}}T}\right)^{1/2}\mathbf{v}_{\lambda\rho}\cdot\mathbf{v}(\mathbf{k},0)\right] \tag{4.11}$$

Inserting (4.11) into (4.7) and (4.8) and using the static correlation formulas, we recover the expressions (3.8), (3.10), and (3.17) (the latter two with $z_{\lambda\rho}$ replaced by ρw_λ), which have been established with the help of kinetic theory. Expressions for the t.c.f.'s containing terms of next to leading order in the wavenumber follow by using the full set (4.1)–(4.4) instead of (4.5)–(4.6). These are discussed in the Appendix.

The correlation functions involving the energy density $\delta\varepsilon$ cannot be compared directly to those found in the previous section, since the latter apply to the kinetic part of the energy density. In lowest order in k the magnetohydrodynamic equations and the static fluctuation formulas yield the following expressions for these t.c.f.'s in leading order in k:

$$S^{\varepsilon n}(\mathbf{k},t)=nk_{\mathrm{B}}T\frac{k^2}{k_{\mathrm{D}}^2}\left[-\frac{1}{nk_{\mathrm{B}}}\left(\frac{\partial P}{\partial T}\right)_n e^{-iz_T t}+\frac{1}{2}\frac{P+\varepsilon}{nk_{\mathrm{B}}T}\sum_{\lambda\rho}\frac{w_\lambda^2-\omega_B^2}{2w_\lambda^2-\omega_p^2-\omega_B^2}\,e^{-iz_{\lambda\rho}t}\right] \tag{4.12}$$

$$S_i^{\varepsilon g}(\mathbf{k},t)=\frac{1}{2}k_{\mathrm{B}}T(P+\varepsilon)\sum_{\lambda\rho}\frac{\rho f_{i,\lambda\rho}}{w_\lambda(2w_\lambda^2-\omega_p^2-\omega_B^2)}\,e^{-iz_{\lambda\rho}t} \tag{4.13}$$

with $f_{i,\lambda\rho}$ given in (3.11)–(3.13), and

$$S^{\varepsilon\varepsilon}(\mathbf{k}, t) = nk_B T^2 c_V e^{-iz_T t} \tag{4.14}$$

The t.c.f.'s (4.12) and (4.13) satisfy a conservation law analogous to (3.37).

The time-dependent correlation functions for the kinetic part of the energy density can be studied as well in the framework of magnetohydrodynamics. In fact, we may define the hydrodynamic analogue of the microscopic kinetic energy density fluctuation as a linear combination of particle density and temperature fluctuations:

$$\delta\varepsilon^{\text{kin,h}}(\mathbf{r}, t) = \frac{3}{2} k_B T\, \delta n(\mathbf{r}, t) + \frac{3}{2} nk_B\, \delta T(\mathbf{r}, t) \tag{4.15}$$

Alternatively we may write:

$$\delta\varepsilon^{\text{kin,h}}(\mathbf{r}, t) = \frac{3}{2} k_B T \left[\frac{1}{nc_V} \left(\frac{\partial P}{\partial T} \right)_n + 1 - \frac{P+\varepsilon}{nc_V T} \right] \delta n(\mathbf{r}, t) + \frac{3}{2} \frac{k_B}{c_V}\, \delta\varepsilon(\mathbf{r}, t) \tag{4.16}$$

As a consequence, t.c.f.'s involving $\delta\varepsilon^{\text{kin,h}}$ can be obtained as linear combinations of known functions. It turns out that we recover in this way the expressions (3.40)–(3.42), which have been derived with the help of kinetic theory. It should be borne in mind, however, that the latter two formulas could be found only for the collision-dominated plasma. The general expressions (3.29) and (3.36) contain dynamic coefficients, which are not encountered in a magnetohydrodynamic treatment.

5. STATIC LIMIT OF THE TIME CORRELATION FUNCTIONS

In the previous sections the leading terms in the small-wavenumber expansions of the time correlation functions have been derived on the basis of kinetic theory and of magnetohydrodynamics. For vanishing t these expressions should reduce to the static correlation functions in the long-wavelength limit.[7,8] In the following we shall verify whether this constraint is satisfied.

In deriving the kinetic expressions for the t.c.f.'s in Section 3 we have concentrated in particular on the asymptotic behavior of these functions for large t. The contributions that dominated this behavior have been obtained by considering the poles that are close to the real axis in the Laplace transforms. Obviously, expressions derived with such a pole approximation are of little use for the discussion of the static behavior. However, an essential simplification in the Laplace transforms of the t.c.f.'s that do not involve the kinetic energy density could be achieved by substituting the long-wavelength expression $z_{\lambda\rho} = \rho w_\lambda$ for the gyro-plasmon poles. In that case expressions like (3.8), which are valid for arbitrary t, can be established. By substituting $t = 0$ in (3.8), and likewise in (3.10) and (3.17) (with the replacements $z_{\lambda\rho} \to \rho w_\lambda$), we arrive at static limiting forms that are consistent with the equilibrium fluctuation formulas for the particle density and the momentum density.

The analytic structure of the Fourier–Laplace transforms of the t.c.f.'s that involve the kinetic energy density is less trivial since these depend on the dynamic coefficients $\bar{b}_i(z)$ and $c(z)$. In leading order in k, (3.28) gives

$$S^{\varepsilon^{\mathrm{kin}} n}(\mathbf{k}, t) = \frac{3}{2} n k_{\mathrm{B}} T \frac{k^2}{k_{\mathrm{D}}^2} \frac{i}{2\pi} \int_{-\infty + i0}^{\infty + i0} dz\, e^{-izt} \left\{ \frac{z(z^2 - \omega_B^2)}{\prod_{\lambda\rho}(z - \rho w_\lambda)} + \left(\frac{2}{3}\right)^{1/2} \omega_p^2 \frac{\bar{b}_1(z)(z^2 - \omega_B^2 \hat{k}_{\|}^2) + \bar{b}_2(z)(z^2 - \omega_B^2)\hat{k}_{\|}^2}{\prod_{\lambda\rho}(z - \rho w_\lambda)[z + ic(z)]} \right\} \tag{5.1}$$

where we have used (2.19) and (2.25). Upon using a partial fraction decomposition, the contribution of the first integral is easily evaluated for arbitrary t. The second integral can be calculated for $t = 0$ by using the inversion formula for the Laplace transform at $t = 0$. If we assume that both $\bar{b}_i(z)$ and $c(z)$ are bounded for $z \to \pm\infty + i0$, we can deduce, by closing the integral contour in the upper half-plane and using the analyticity of $\bar{b}_i(z)$ and $[z + ic(z)]^{-1}$ in that region, that the second integral vanishes for $t = 0$. Hence we have found

$$S^{\varepsilon^{\mathrm{kin}} n}(\mathbf{k}, t = 0) = \frac{3}{2} n k_{\mathrm{B}} T \frac{k^2}{k_{\mathrm{D}}^2} \tag{5.2}$$

in agreement with the static formula.

The expressions (3.31) for the t.c.f.'s of the kinetic energy density and the momentum density can likewise be employed to derive static results in the long-wavelength limit:

$$S_i^{\varepsilon^{\mathrm{kin}} g}(\mathbf{k}, t = 0) = 0 \tag{5.3}$$

for $i = \|, \perp$, and t.

Finally, we consider the autocorrelations of the kinetic energy density. From (3.39) we obtain for vanishing wavenumber

$$S^{\varepsilon^{\mathrm{kin}} \varepsilon^{\mathrm{kin}}}(\mathbf{k}, t) = \frac{3}{2} n (k_{\mathrm{B}} T)^2 \frac{i}{2\pi} \int_{-\infty + i0}^{\infty + i0} dz \frac{e^{-izt}}{z + ic(z)} \tag{5.4}$$

where we used (2.19), (2.24), and (2.25). Writing $(z + ic)^{-1}$ as the sum of

z^{-1} and $-ic[z(z+ic)]^{-1}$, and assuming as before $c(z)$ to be bounded for $z \to \pm\infty + i0$, we prove easily

$$S^{\varepsilon^{\rm kin}\varepsilon^{\rm kin}}(\mathbf{k}, t=0) = \frac{3}{2} n(k_{\rm B} T)^2 \tag{5.5}$$

which corroborates the static fluctuation formula.

It should be noted that an erroneous result would have been found by substituting directly $t=0$ in (3.40). Indeed, the latter is an asymptotic expression which is valid for large t only. The fact that asymptotic expressions for time correlation functions involving the kinetic energy may yield wrong results at $t=0$ has been noted before, both for neutral fluids[9] and for unmagnetized plasmas.[6] It is related to the lack of conservation of the kinetic energy, as we shall see below.

Let us turn now to the magnetohydrodynamic expressions for the correlation functions. In the derivation of Section 4 the static fluctuation formulas are used as initial conditions. As a consequence, these formulas are recovered necessarily by substituting $t=0$ in the magnetohydrodynamic expressions. In particular, we may check that (4.12)–(4.14) yield for $t=0$

$$S^{\varepsilon n}(\mathbf{k}, t=0) = 3nk_{\rm B} T\left(\frac{1}{nk_{\rm B} T\kappa_T} - \frac{1}{2}\right)\frac{k^2}{k_{\rm D}^2} \tag{5.6}$$

$$S_i^{\varepsilon g}(\mathbf{k}, t=0) = 0 \tag{5.7}$$

$$S^{\varepsilon\varepsilon}(\mathbf{k}, t=0) = nk_{\rm B} T^2 c_V \tag{5.8}$$

In Section 4 we also considered correlation functions involving the fluctuation of the magnetohydrodynamic kinetic energy, as defined in (4.15). It was found that the t.c.f.'s for this kinetic energy fluctuation have the same form as (3.40)–(3.42). As the latter are now interpreted as magnetohydrodynamic formulas, we may substitute $t=0$ so as to get

$$S^{\varepsilon^{\rm kin,h} n}(\mathbf{k}, t=0) = \frac{3}{2} nk_{\rm B} T \frac{k^2}{k_{\rm D}^2} \tag{5.9}$$

$$S_i^{\varepsilon^{\rm kin,h} n}(\mathbf{k}, t=0) = 0 \tag{5.10}$$

$$S^{\varepsilon^{\rm kin,h}\varepsilon^{\rm kin,h}}(\mathbf{k}, t=0) = \frac{9}{4} n(k_{\rm B} T)^2 \frac{k_{\rm B}}{c_V} \tag{5.11}$$

Comparison with (5.2)–(5.5) shows that for $t=0$ the magnetohydrodynamic t.c.f. for the autocorrelations of the kinetic energy density differs from its kinetic counterpart. For large t, however, these functions do agree, as we have seen in the previous section. This somewhat peculiar state of affairs can be understood by returning to the microscopic definition of the kinetic energy density fluctuation in Fourier space:

$$\delta\varepsilon^{\mathrm{kin}}(\mathbf{k}) = \sum_{\alpha} \frac{p_\alpha^2}{2m} \exp(-i\mathbf{k}\cdot\mathbf{r}_\alpha) \tag{5.12}$$

In kinetic theory one treats the autocorrelations of this microscopic quantity. The magnetohydrodynamic kinetic energy fluctuation (4.16) may likewise be given as a microscopic quantity, in terms of the fluctuations of the microscopic particle density and the microscopic total energy density,[7]

$$\delta n(\mathbf{k}) = \sum_{\alpha} \exp(-i\mathbf{k}\cdot\mathbf{r}_\alpha) \tag{5.13}$$

$$\delta\varepsilon(\mathbf{k}) = \delta\varepsilon^{\mathrm{kin}}(\mathbf{k}) - \frac{1}{2V} \sum_{\mathbf{q}(\neq 0, \neq \mathbf{k})} \frac{e^2\mathbf{q}\cdot(\mathbf{k}-\mathbf{q})}{q^2(\mathbf{k}-\mathbf{q})^2} \sum_{\alpha\neq\beta} \exp(i\mathbf{q}\cdot\mathbf{r}_{\alpha\beta} - i\mathbf{k}\cdot\mathbf{r}_\alpha) \tag{5.14}$$

The kinetic energy density fluctuation $\delta\varepsilon^{\mathrm{kin}}$ can be written now as the sum of a linear combination of δn, $\delta\varepsilon$, and a remainder $\delta\bar{\varepsilon}^{\mathrm{kin}}$ that is "orthogonal" to both δn and $\delta\varepsilon$, i.e.,

$$\frac{1}{V}\langle [\delta n(\mathbf{k})]^* \, \delta\bar{\varepsilon}^{\mathrm{kin}}(\mathbf{k})\rangle = 0 \tag{5.15}$$

$$\frac{1}{V}\langle [\delta\varepsilon(\mathbf{k})]^* \, \delta\bar{\varepsilon}^{\mathrm{kin}}(\mathbf{k})\rangle = 0 \tag{5.16}$$

Upon evaluating $\delta\bar{\varepsilon}^{\mathrm{kin}}$, it is found that the linear combination of δn and $\delta\varepsilon$ occurring in $\delta\varepsilon^{\mathrm{kin}}$ is precisely the "hydrodynamic" fluctuation (4.16) with (5.13) and (5.14) inserted, so that we have

$$\delta\varepsilon^{\mathrm{kin}}(\mathbf{k}) = \delta\varepsilon^{\mathrm{kin,h}}(\mathbf{k}) + \delta\bar{\varepsilon}^{\mathrm{kin}}(\mathbf{k}) \tag{5.17}$$

As a function of time, $\delta\varepsilon^{\mathrm{kin,h}}$ will change rather slowly, since it is a linear combination of quantities that are conserved in the long-wavelength limit. In contrast, $\delta\bar{\varepsilon}^{\mathrm{kin}}$ may decay rapidly through relaxation processes. Although $\delta\varepsilon^{\mathrm{kin}}$ and $\delta\varepsilon^{\mathrm{kin,h}}$ are different for $t=0$, they will tend to the same limit as t goes to infinity. Correspondingly, the t.c.f.'s involving these quantities will in general be different for $t=0$, but they must agree for $t\to\infty$. A particular case is the cross-correlation function of the kinetic energy density and the particle density. As a consequence of (5.15), one has

$$S^{\varepsilon^{\mathrm{kin}}n}(\mathbf{k}, t) = S^{\varepsilon^{\mathrm{kin,h}}n}(\mathbf{k}, t) \tag{5.18}$$

both for large t and for $t=0$. For intermediate values of t the two functions do not necessarily coincide.

APPENDIX. HIGHER-ORDER CONTRIBUTIONS IN THE TIME CORRELATION FUNCTIONS

In Sections 3 and 4 we have studied the asymptotic form of the dynamic structure factor $S^{nn}(\mathbf{k}, t)$ for large t. The result (3.7), valid for small k, contains contributions from the four gyro-plasmon modes. The weights of each of these exponentially decaying contributions has been determined in leading (second) order of k. In this Appendix we will show how the terms of order k^4 can be obtained from kinetic theory. It will turn out that in that order the thermal mode contributes as well. Moreover, the weights of the gyro-plasmon contributions will be found to depend on dynamic coefficients. Finally, we will give the general form of the t.c.f.'s S_i^{ng} and S_i^{gg} up to order k^2.

To derive the higher-order terms in $S^{nn}(\mathbf{k}, t)$ we shall start by giving explicit expressions for the damping and dispersion functions (2.26) which determine the gyro-plasmon mode frequencies (2.25) in order k^2. From ref. 2 we obtain the following formulas for $N_{1\lambda}^{\rho}$ and $N_{2\lambda}$:

$$nk_B T N_{1\lambda}^{\rho} = -\rho w_\lambda^3(\omega_B H_1 - \kappa_T^{-1}) - i w_\lambda^2(\omega_B^2 H_2 + \omega_p^2 H_3) + \rho w_\lambda \omega_B \hat{k}_{\|}^2(\omega_p^2 H_4 - \omega_B \kappa_T^{-1}) + i\omega_p^2 \omega_B^2 \hat{k}_{\|}^2 H_5 \quad \text{(A.1)}$$

$$N_{2\lambda} = w_\lambda^2[\omega_p^2(\bar{b}_1 + \bar{b}_2 \hat{k}_{\|}^2)^2 + \omega_B^2 \hat{k}_{\perp}^2 \bar{b}_1^2] - \omega_p^2 \omega_B^2 \hat{k}_{\|}^2[\bar{b}_1^2 \hat{k}_{\perp}^2 + (\bar{b}_1 + \bar{b}_2)^2 \hat{k}_{\|}^2] \quad \text{(A.2)}$$

The functions $H_i(\hat{k}_{\|}^2, z)$ in (A.1) are polynomials in $\hat{k}_{\|}^2$ that depend on the seven anisotropic viscosity coefficients $\eta_1(z),\ldots, \eta_5(z)$, $\eta_V(z)$, and $\zeta(z)$ of a magnetized plasma:

$$H_1 = -2(\eta_4 + \eta_5)\hat{k}_{\|}^2 + 2\eta_4 \quad \text{(A.3)}$$

$$H_2 = \left(\frac{5}{3}\eta_1 - 4\eta_2 + 2\eta_3 - \eta_V + 2\zeta\right)\hat{k}_{\|}^2 - \frac{5}{3}\eta_1 + 4\eta_2 + \eta_V - 2\zeta \quad \text{(A.4)}$$

$$H_3 = 2(\eta_1 + \eta_2 - 2\eta_3)\hat{k}_{\|}^4 + 2(-2\eta_2 + 2\eta_3 + 3\zeta)\hat{k}_{\|}^2 - \frac{2}{3}\eta_1 + 2\eta_2 + \eta_V - 2\zeta \quad \text{(A.5)}$$

$$H_4 = -2(\eta_4 + 2\eta_5)\hat{k}_{\|}^2 + 2(\eta_4 + \eta_5) \quad \text{(A.6)}$$

$$H_5 = (3\eta_1 - 4\eta_2 + \eta_3 + 6\zeta)\hat{k}_{\|}^2 - \frac{5}{3}\eta_1 + 4\eta_2 + \eta_3 + \eta_V - 2\zeta \quad \text{(A.7)}$$

The dynamic viscosity coefficients are determined by the cylinder-symmetric decomposition of the matrix element

$$\langle i|\varphi^{c}|j\rangle + \left\langle i\left|\Sigma\bar{Q}\frac{1}{z-\bar{Q}\Sigma\bar{Q}}\bar{Q}\Sigma\right|j\right\rangle \qquad (i, j=1, 2, 3) \qquad \text{(A.8)}$$

[cf. (2.13) and (2.14)], as given in ref. 2.

The dynamic structure factor up to order k^4 follows from (3.1) with (3.4) by retaining the next to leading terms both in the minor $M_{00}(\mathbf{k}, z)$ and in the determinant $\Delta(\mathbf{k}, z)$. The latter has been given already in (2.18) with (2.19) and (2.25). The minor reads, up to second order in k,

$$\begin{aligned} M_{00}(\mathbf{k}, z) = {} & \frac{2c_V}{3k_B} z(z^2-\omega_B^2)(z-z_T) \\ & + \frac{ik^2}{nm}(z+ic)(z^2\bar{H}_1 + iz\omega_B\bar{H}_2 + \omega_B^2\bar{H}_3) \\ & + v_0^2k^2\{-z^2[\bar{b}_1^2 + (2\bar{b}_1+\bar{b}_2)\bar{b}_2\hat{k}_{\|}^2] + \omega_B^2(\bar{b}_1+\bar{b}_2)^2\hat{k}_{\|}^2\} \end{aligned} \qquad \text{(A.9)}$$

The functions $\bar{H}_i(\hat{k}_{\|}^2, z)$ are

$$\bar{H}_1 = H_5 \qquad \text{(A.10)}$$

$$\bar{H}_2 = -H_1 \qquad \text{(A.11)}$$

$$\bar{H}_3 = H_2 - H_5 \qquad \text{(A.12)}$$

Upon dividing $M_{00}(\mathbf{k}, z)$ by $\Delta(\mathbf{k}, z)$, we obtain the asymptotic form for $S^{nn}(\mathbf{k}, t)$ by using the pole approximation, as discussed in Section 3. The thermal mode then contributes through the last term of (A.9). Introducing the velocity of sound c_s through the thermodynamic relation

$$c_s^2 = \frac{Tc_P}{n^2mc_V(c_P-c_V)}\left(\frac{\partial P}{\partial T}\right)_n^2 \qquad \text{(A.13)}$$

with c_P the isobaric specific heat per particle, we find the thermal mode contribution to $S^{nn}(\mathbf{k}, t)$ as

$$\frac{k^4c_s^2}{k_D^2\omega_p^2}\left(1-\frac{c_V}{c_P}\right)e^{-iz_Tt} \qquad \text{(A.14)}$$

The derivation of the contributions of the gyro-plasmon modes is rather more laborious. Taking the residues, we encounter derivatives $\partial D_{\lambda\rho}/\partial z$ at $z=\rho w_\lambda$. These can be evaluated explicitly by using (2.26), (A.1), and (A.2). As a consequence, derivatives of the dynamical viscosities contained in H_i

and of the dynamical coefficients $\bar{b}_i(z)$ and $c(z)$ show up. We prefer to present the result for $S^{nn}(\mathbf{k}, t)$ in the form which still contains $D_{\lambda\rho}$ and its derivative:

$$
\begin{aligned}
S^{nn}(\mathbf{k}, t) = {} & \frac{k^4 c_s^2}{k_{\mathrm{D}}^2 \omega_p^2}\left(1 - \frac{c_V}{c_P}\right) e^{-iz_T t} \\
& + \frac{1}{2}\frac{k^2}{k_{\mathrm{D}}^2} \sum_{\lambda\rho} \frac{\rho}{w_\lambda(2w_\lambda^2 - \omega_p^2 - \omega_B^2)} \\
& \times \left\{ \rho w_\lambda (w_\lambda^2 - \omega_B^2)\left(1 + ik^2 \mathscr{D}_{\lambda\rho} - \frac{k^2}{nk_{\mathrm{B}} T \kappa_T k_{\mathrm{D}}^2}\right) \right. \\
& + ik^2(-3w_\lambda^2 + \omega_B^2) D_{\lambda\rho} + \frac{ik^2}{nm}(w_\lambda^2 \bar{H}_1 + i\rho w_\lambda \omega_B \bar{H}_2 + \omega_B^2 \bar{H}_3) \\
& \left. + v_0^2 k^2 \frac{-w_\lambda^2[\bar{b}_1^2 + (2\bar{b}_1 + \bar{b}_2)\bar{b}_2 \hat{k}_{\|}^2] + \omega_B^2(\bar{b}_1 + \bar{b}_2)^2 \hat{k}_{\|}^2}{\rho w_\lambda + ic} \right\} e^{-iz_{\lambda\rho} t}
\end{aligned} \tag{A.15}
$$

with

$$
\begin{aligned}
\mathscr{D}_{\lambda\rho} = {} & \frac{\rho}{2w_\lambda}(D_{\lambda\rho} - D_{\lambda,-\rho}) \\
& + \frac{\rho}{2w_\lambda^2 - \omega_p^2 - \omega_B^2}\,[2w_\lambda D_{\lambda\rho} - w_\lambda(D_{-\lambda\rho} + D_{-\lambda,-\rho}) \\
& - w_{-\lambda}(D_{-\lambda\rho} - D_{-\lambda,-\rho})] - \frac{\partial D_{\lambda\rho}}{\partial z}
\end{aligned} \tag{A.16}
$$

In all z-dependent functions, i.e. $D_{\lambda\rho}$, $\partial D_{\lambda\rho}/\partial z$, $\bar{b}_i$, c, and $\bar{H}_i$, one should substitute $z = \rho w_\lambda$. The term containing the isothermal compressibility arises from the static structure factor $1 + nh(k)$, which reads, up to fourth order,[7,8]

$$
1 + nh(k) = \frac{k^2}{k_{\mathrm{D}}^2} - \frac{1}{nk_{\mathrm{B}} T \kappa_T} \frac{k^4}{k_{\mathrm{D}}^4} + \cdots \tag{A.17}
$$

The result (A.15) is rather complicated. Its main features can be summarized as follows. In fourth order $S^{nn}(\mathbf{k}, t)$ contains contributions both from the heat mode and from the gyro-plasmon modes. The amplitude of the heat mode term is determined by thermodynamic quantities. The amplitudes of the gyro-plasmon mode contributions depend on dynamic coefficients evaluated at the fundamental gyro-plasmon frequencies ρw_λ.

These dynamic coefficients can all be expressed in matrix elements of the frequency matrix, as given in (2.13), (2.14), and (A.8). These matrix elements are also encountered in the damping and dispersion parts of the mode frequencies.

The t.c.f.'s which describe the cross-correlations of the particle density and the components of the momentum density, S_i^{ng} and S_i^{gg}, can also be obtained up to order k^2 from kinetic theory. We give here only the general form of these t.c.f.'s:

$$S_i^{ng} = \frac{1}{2} nk_{\mathrm{B}} T \sum_{\lambda\rho} \frac{\rho F_{i,\lambda\rho}}{w_\lambda(2w_\lambda^2 - \omega_p^2 - \omega_B^2)} e^{-iz_{\lambda\rho}t} \qquad (i = \|, \perp, t) \quad \text{(A.18)}$$

$$S_i^{gg} = \frac{1}{2} nmk_{\mathrm{B}} T \sum_{\lambda\rho} \frac{\rho G_{i,\lambda\rho}}{w_\lambda(2w_\lambda^2 - \omega_p^2 - \omega_B^2)} e^{-iz_{\lambda\rho}t} \qquad (i = 1,..., 6) \quad \text{(A.19)}$$

In second order of k the heat mode does not contribute yet. The functions $F_{i,\lambda\rho}$ and $G_{i,\lambda\rho}$, which depend on $z = \rho w_\lambda$, have the form

$$\begin{aligned} F_{i,\lambda\rho} &= f_{i,\lambda\rho}(1 + ik^2 \mathscr{D}_{\lambda\rho}) + ik^2 \mathscr{F}_{i,\lambda\rho} D_{\lambda\rho} \\ &\quad + \frac{ik^2}{nm} \mathscr{H}_{i,\lambda\rho} + v_0^2 k^2 \frac{\mathscr{B}_{i,\lambda\rho}}{\rho w_\lambda + ic} \end{aligned} \qquad \text{(A.20)}$$

$$\begin{aligned} G_{i,\lambda\rho} &= \rho w_\lambda g_{i,\lambda\rho}(1 + ik^2 \mathscr{D}_{\lambda\rho}) + ik^2 \mathscr{G}'_{i,\lambda\rho} D_{\lambda\rho} \\ &\quad + \frac{k^2}{nk_{\mathrm{B}} T \kappa_T k_{\mathrm{D}}^2} \mathscr{G}''_{i,\lambda\rho} + \frac{ik^2}{nm} \mathscr{H}_{i,\lambda\rho} + v_0^2 k^2 \frac{\mathscr{B}_{i,\lambda\rho}}{\rho w_\lambda + ic} \end{aligned} \qquad \text{(A.21)}$$

The coefficients $\mathscr{F}_{i,\lambda\rho}$, $\mathscr{G}'_{i,\lambda\rho}$, and $\mathscr{G}''_{i,\lambda\rho}$ are simple functions of ρw_λ, ω_p and ω_B. Furthermore, $\mathscr{H}_{i,\lambda\rho}$ are linear combinations of the seven dynamical viscosity coefficients. Finally, $\mathscr{B}_{i,\lambda\rho}$ are quadratic functions of the dynamic coefficients $\bar{b}_i$. All z-dependent functions are to be evaluated at $z = \rho w_\lambda$.

As a special case we may consider the collision-dominated regime. It may be verified that the kinetic expressions (A.15), (A.18), and (A.19) reduce in that case to the formulas that can be derived from the magnetohydrodynamic equations of Section 4.

ACKNOWLEDGMENTS

This investigation is part of the research programme of the Stichting voor Fundamenteel Onderzoek der Materie, which is financially supported by the Nederlandse Organisatie voor Wetenschappelijk Onderzoek.

REFERENCES

1. B. Jancovici, N. Macris, and Ph. A. Martin, *J. Stat. Phys.* **47**:229 (1987).
2. L. G. Suttorp and J. S. Cohen, *Physica* **133A**:370 (1985).
3. L. G. Suttorp and A. J. Schoolderman, *Physica* **141A**:1 (1987).
4. M. C. Marchetti, T. R. Kirkpatrick, and J. R. Dorfman, *J. Stat. Phys.* **46**:679, **49**:871 (1987).
5. D. Forster and P. C. Martin, *Phys. Rev. A* **2**:1575 (1970); D. Forster, *Phys. Rev. A* **9**:943 (1974).
6. M. Baus, *Physica* **79A**:377 (1975).
7. L. G. Suttorp and J. S. Cohen, *Physica* **133A**:357 (1985).
8. P. Vieillefosse, *J. Stat. Phys.* **41**:1015 (1985).
9. P. Résibois, *Physica* **59**:587 (1972).